# CRAMÉR-RAO BOUND FOR RANGE ESTIMATION

*Yiyin Wang, Geert Leus and Alle-Jan van der Veen*

Delft University of Technology, Fac. EEMCS, Mekelweg 4, 2628CD Delft, The Netherlands

**ABSTRACT**

In this paper, we derive the Cramér-Rao bound (CRB) for range estimation, which does not only exploit the range information in the time delay, but also in the amplitude of the received signal. This new bound is lower than the conventional CRB that only makes use of the range information in the time delay. We investigate the new bound in an additive white Gaussian noise (AWGN) channel with attenuation by employing both narrowband (NB) signals and ultra-wideband (UWB) signals. For NB signals, the new bound can be 3dB lower than the conventional CRB under certain conditions. However, there is not much difference between the new bound and the conventional CRB for UWB signals. Further, shadowing effects are added into the data model. Several CRB-like bounds for range estimation are derived to take these shadowing effects into account.

***Index Terms***— Cramér-Rao bound, ranging, shadowing

## 1. INTRODUCTION

The Cramér-Rao bound (CRB) is a standard benchmark to evaluate the performance of an estimator. In this paper, we investigate the CRB for range ($D$) estimation (the distance between the transmitter and the receiver), which is an important parameter for localization. There are two elementary measurements for range estimation: the received signal strength (RSS) and the time of arrival (TOA–$\tau$) [1]. The existing literature mostly treats them separately to derive the CRB(D), e.g., see [1][2]. Some of them [3] insert the path-loss model (indicating the relationship between the RSS and the range) into the received signal-to-noise ratio (SNR), which is viewed as a parameter of the CRB($\tau$). However, they do not exploit the range information in the RSS and the TOA jointly. Previous work in [4] proposes to use both the RSS and the TOA to improve the ranging accuracy and derives a CRB(D) by simply, yet incorrectly, assuming that they are uncorrelated. Moreover, it does not provide a method to combine them. Other joint methods [5] are for location estimation. They propose to fuse the TOA measurements and the RSS measurements to get a lower bound for localization, meanwhile they assume the estimation error of the TOA has constant variance regardless of its distance dependency.

We investigate the relationship among the RSS, the TOA and the range $D$, and use both the RSS and the TOA for the CRB($D$). We do not use the RSS explicitly as a parameter, but the amplitude of the received signal. Single path propagation is assumed, yet the path-loss model is taken into account in the received signal model to explore the distance dependency of the amplitude of the received signal. The amplitude and the TOA are represented as a function of $D$ explicitly in the received signal model. Thus, we can derive the CRB($D$) directly.

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The rest of the paper is organized as follows. In Section 2, we derive the true CRB($D$) in an additive white Gaussian noise (AWGN) channel with attenuation. Some results are shown. Further, the maximum likelihood estimator (MLE) of $D$ is proposed. In Section 3, we include shadowing. Due to the difficulty to derive the true CRB(D) directly in this case, we propose several CRB-like bounds and suboptimal estimators for $D$. Numerical results are also shown in Section 3. We conclude the paper in Section 4.

## 2. CRB AND ESTIMATOR IN AN AWGN CHANNEL WITH ATTENUATION

The received signal through an AWGN channel with an attenuation coefficient is

$$r(t) = \alpha s(t-\tau) + n(t). \tag{1}$$

where $\tau$ is the unknown deterministic time delay, which is related to the range $D$ as $\tau = D/c$, with $c$ the signal propagation speed. The channel attenuation coefficient $\alpha$ is related to $D$ following the distance-power law [6] as $\alpha = k_0/\sqrt{D^{\gamma_0}}$, with $k_0$ a constant parameter and $\gamma_0$ also a constant depending on the environment, e.g., in free space $\gamma_0 = 2$. The transmitted waveform is represented by $s(t)$. We use $n(t)$ to denote AWGN with double-sided power spectral density $N_0/2$, which is filtered out by an ideal anti-aliasing filter of bandwidth $B$. We assume $B$ is larger than the signal bandwidth. The continuous signal is sampled at the Nyquist rate $2B = 1/T_s$. The received data samples are collected in a vector $\mathbf{r} = [r(0), r(T_s), r(2T_s), \ldots, r((N-1)T_s)]^T$, which can be written as

$$\mathbf{r} = k_0 D^{-\frac{\gamma_0}{2}} \mathbf{s}_D + \mathbf{n}. \tag{2}$$

where $\mathbf{s}_D = [s(-\tau), s(T_s-\tau), \ldots, s((N-1)T_s-\tau)]^T$ and $\mathbf{n}$ contains noise samples with variance $\sigma^2 = N_0 B$. In this model, we find the range information not only in the time delay, but also in the amplitude of the received signal. Therefore, the CRB($D$) based on the data model (2) is

$$\text{CRB}(D) \quad = \quad \left\{ E_{\mathbf{r}} \left[ -\frac{\partial^2}{\partial D^2} \ln(p(\mathbf{r}; D)) \right] \right\}^{-1}. \tag{3}$$

with $p(\mathbf{r}; D)$ following a Gaussian distribution [7]. Assuming the observation window includes the whole waveform, it leads to

$$\text{CRB}(D) \quad = \quad \frac{c^2 D^{\gamma_0+2}}{2k_0^2 \frac{\mathcal{E}}{N_0} \left( \frac{\gamma_0^2 c^2}{4} + D^2 \overline{F^2} + \gamma_0 c D \overline{F'^2} \right)}. \tag{4}$$

where $\mathcal{E} = \int_0^{T_o} s^2(t)dt$ is the transmitted signal energy, $\overline{F^2} = \int_{-\infty}^{\infty} (2\pi F)^2 |S(F)|^2 dF \Big/ \int_{-\infty}^{\infty} |S(F)|^2 dF$ is the mean square bandwidth of the signal, with $S(F)$ the Fourier transform of $s(t)$, and $\overline{F'^2} = \int_0^{T_o} s(t)\frac{ds(t)}{dt}dt \Big/ \int_0^{T_o} s^2(t)dt = \frac{s^2(t)}{2}\Big|_0^{T_o} \Big/ \mathcal{E}$. Since

$s(t)$ is smooth in $[0, T_o)$, we may assume $s(0) = s(T_o)$. As a result, $\overline{F'^2} = 0$. Therefore, the CRB($D$) in (4) can be simplified as

$$\text{CRB}(D) \quad = \quad \frac{c^2 D^{\gamma_0+2}}{2k_0^2 \frac{\mathcal{E}}{N_0}\left(\frac{\gamma_0^2 c^2}{4} + D^2\overline{F^2}\right)}. \qquad (5)$$

We now compare CRB($D$) in (5) with the results in [3]. We first apply the method in [3] to derive the CRB($\tau$), and then obtain the CRB($D$) as a transformation of the CRB($\tau$). It results into

$$\text{CRB}_{ref}(D) = \frac{c^2 D^{\gamma_0}}{2k_0^2 \frac{\mathcal{E}}{N_0}\overline{F^2}}. \qquad (6)$$

Though the method in [3] takes the path-loss model into account, it only exploits the range information in the time delay. The CRB($D$) in (5) has one more positive term $\gamma_0^2 c^2/4$ in the denominator, as a result of additionally investigating the range information in the amplitude of the received signal. Setting $\gamma_0^2 c^2/4 = D^2\overline{F^2}$, we can obtain a critical distance $D_c = \gamma_0 c/(2\sqrt{\overline{F^2}})$, which is determined by the environment, the signal propagation speed and the mean square bandwidth of the transmitted signal. When $D \gg D_c$, CRB($D$) $\approx$ CRB$_{ref}(D)$. On the other hand, when $D \approx D_c$, we can gain about 3dB by taking the additional information into account.

Now, we will give some examples. It is known that $\mathcal{E} = \int_0^{T_o} s^2(t)dt = T_o\int_{-\infty}^{\infty}\Phi(F)dF$, where $\Phi(F)$ is the power spectral density of the signal. Assuming $\Phi(F)$ is uniformly distributed over the bandwidth ($W = f_H - f_L$) of the signal, we get $\overline{F^2} = 4\pi^2(\frac{W^2}{3} + f_H f_L)$. A larger bandwidth and a higher central frequency result in a larger $\overline{F^2}$ and a lower CRB. Using $k_0 = 1$, $\gamma_0 = 2$, $c = 3 \cdot 10^8$m/s and the whole bandwidth ($W = 10.6\text{GHz} - 3.1\text{GHz}$) of the ultra-wide band (UWB) signals, $D_c$ is approximately 6.6mm, which is quite small. Hence, for an indoor environment, where $D$ is in the range of a few meters, we do not need the new method and consider the range information in both the amplitude and the delay. This is due to the large bandwidth and the high central frequency of the UWB signals. However, if we only use a narrowband (NB) signal, for example, with a bandwidth from 0 to 5MHz, $D_c$ would be 16.5m under those circumstances. We can then benefit from the new method for an indoor environment.

The left half of Table 1 shows the CRB($D$) (5) in an AWGN channel with attenuation employing UWB signals with different bandwidths. The transmitted power spectral density of the signal $\Phi(F)$ is restricted below the FCC mask ($-41.3$ dBm/MHz). Further, $k_0 = 1, \gamma_0 = 2, c = 3 \cdot 10^8$m/s$, N_0 = 2 \cdot 10^{-20}$w/Hz, and $T_o = 100$ns. Clearly, increasing the bandwidth and central frequency, we obtain a more accurate range estimation. Meanwhile, the right half of Table 1 compares the new CRB($D$) (5) with the conventional CRB$_{ref}(D)$ (6) employing NB signals. The NB signals have a frequency range from 0 to 5MHz. The observation time $T_o$ is 1$\mu$s. Other parameters are kept the same. We can see the new method is much more accurate than the conventional method for the NB signals in the relevant range. The new CRB($D$) (5) is almost 3dB lower than the conventional CRB$_{ref}(D)$ (6).

The MLE of $D$ is derived, which can asymptotically attain the bound in (5). We would like to find the $D$ that maximizes $p(\mathbf{r}; D)$ or ln $p(\mathbf{r}; D)$, leading to

$$\hat{D} \quad = \quad \underset{0<D<cT_o}{\text{argmin}}\left\{k_0 D^{-\gamma_0}\mathcal{E} - 2D^{-\frac{\gamma_0}{2}}\mathcal{E}_{rs}(D)\right\}. \qquad (7)$$

where $\mathcal{E}_{rs}(D)$ is a function of $D$: $\mathcal{E}_{rs}(D) = \int_0^{T_o} r(t)s(t - D/c)dt$. A grid search is then executed to look for $\hat{D}$. The variance of this estimator approaches (5) as long as the data record is large enough.

## 3. CRBS AND ESTIMATORS IN AN AWGN CHANNEL WITH ATTENUATION AND SHADOWING

When we include shadowing effects, which represent a more realistic environment, the received signal is

$$r(t) \quad = \quad \tilde{\alpha}Xs(t-\tau) + n(t). \qquad (8)$$

where X is a random variable modeling the shadowing effects and following a lognormal distribution $20\log_{10}X \sim \mathcal{N}(0, \sigma_x^2)$, and $\tilde{\alpha}$ still follows a distance-power law as $\tilde{\alpha} = \tilde{k_0}^2/\sqrt{D^{\gamma_0}}$. In order to get a fair comparison, we normalize the average channel energy, resulting in $\tilde{\alpha}^2 E[X^2] = \alpha^2$. Therefore, $\tilde{k_0}^2 = k_0^2/E[X^2]$. The normalization excludes the influence of the shadowing coefficient to the average received energy. The definitions for other parameters remain the same as in the last section. The received signal vector is

$$\mathbf{r} \quad = \quad \tilde{k}_0 D^{-\frac{\gamma_0}{2}} X\mathbf{s}_D + \mathbf{n}. \qquad (9)$$

Since $X$ is independent of $D$, it can be viewed as a nuisance parameter. The MLE for $D$ tries to find the $D$ that maximizes $p(\mathbf{r}; D)$. It is known that $p(\mathbf{r}; D) = \int p(\mathbf{r}|X; D)p(X)dX$. By integration, we can get rid of $X$ leading to a closed form of $p(\mathbf{r}; D)$, which is not only needed for the MLE, but also for the CRB($D$). However, the integration is very difficult to derive in closed form.

Due to the difficulties to obtain the closed form of $p(\mathbf{r}; D)$, we will derive the CRB for $\boldsymbol{\theta} = [D, X]^T$. Since we have prior knowledge of $X$, the Bayesian information matrix (BIM) [7] for $\boldsymbol{\theta}$ is employed

$$\mathbf{I}_B(\boldsymbol{\theta}) \quad = \quad E_{\boldsymbol{\theta}}[\mathbf{I}_D(\boldsymbol{\theta})] + \mathbf{I}_P(\boldsymbol{\theta}), \qquad (10)$$

$$[\mathbf{I}_D(\boldsymbol{\theta})]_{ij} \quad = \quad -E_{\mathbf{r}|\boldsymbol{\theta}}\left[\frac{\partial^2}{\partial\boldsymbol{\theta}_i\partial\boldsymbol{\theta}_j}\text{ln}p(\mathbf{r}|\boldsymbol{\theta})\right], \qquad (11)$$

$$[\mathbf{I}_P(\boldsymbol{\theta})]_{ij} \quad = \quad -E_{\boldsymbol{\theta}}\left[\frac{\partial^2}{\partial\boldsymbol{\theta}_i\partial\boldsymbol{\theta}_j}\text{ln}p(\boldsymbol{\theta})\right]. \qquad (12)$$

where $\mathbf{I}_D(\boldsymbol{\theta})$ represents information obtained from the data, $\mathbf{I}_P(\boldsymbol{\theta})$ indicates the prior information and $p(\mathbf{r}|\boldsymbol{\theta})$ follows a Gaussian distribution. Hence, we obtain

$$\mathbf{I}_D(\boldsymbol{\theta}) = \frac{2\tilde{k_0}^2}{D^{\gamma_0}}\frac{\mathcal{E}}{N_0}\begin{bmatrix} X^2(\frac{\gamma_0^2}{4D^2} + \frac{\overline{F^2}}{c^2} + \frac{\gamma_0\overline{F'^2}}{cD}) & -X(\frac{\gamma_0}{2D} + \frac{\overline{F'^2}}{c}) \\ -X(\frac{\gamma_0}{2D} + \frac{\overline{F'^2}}{c}) & 1 \end{bmatrix},$$

$$\mathbf{I}_P(\boldsymbol{\theta}) = \begin{bmatrix} 0 & 0 \\ 0 & X_c \end{bmatrix}, \quad \text{where } X_c = E_X\left[-\frac{\partial^2\text{ln}p(X)}{\partial X^2}\right].$$

We now have all the ingredients to derive several bounds. The first bound is the Hybrid CRB (HCRB) [8], which is HCRB($D$) = $\left[\mathbf{I}_B^{-1}(\boldsymbol{\theta})\right]_{11}$ as shown in (13), with $M_2 = E[X^2]$ and $M_1 = E[X]$. It covers the case where the desired deterministic parameter and random nuisance parameters are jointly estimated, and it is a bound for the estimators that take the prior knowledge of $X$ into account.

The second bound is the Modified CRB (MCRB) [9]: MCRB($D$) = $1/[\mathbf{I}_B(\boldsymbol{\theta})]_{11}$, which is

$$\text{MCRB}(D) = \frac{c^2 D^{\gamma_0+2}}{E[X^2]2\tilde{k_0}^2\frac{\mathcal{E}}{N_0}\left(\frac{\gamma_0^2 c^2}{4} + D^2\overline{F^2} + \gamma_0 cD\overline{F'^2}\right)}. \qquad (14)$$

| | UWB signals $\sigma_D$(m) | | | | NB signals, 0MHz $\sim$ 5MHz | |
|---|---|---|---|---|---|---|
| $D$(m) | 3.1GHz $\sim$ 10.6GHz | 7.316GHz $\sim$ 8.684GHz | 3.658GHz $\sim$ 4.342GHz | $D$(m) | $\sigma_D$(m) | $\sigma_{Dref}$(m) |
| 1 | $9.0786 \cdot 10^{-8}$ | $1.9066 \cdot 10^{-7}$ | $5.3924 \cdot 10^{-7}$ | 10 | $1.4317 \cdot 10^{-2}$ | $2.7671 \cdot 10^{-2}$ |
| 3 | $2.7236 \cdot 10^{-7}$ | $5.7199 \cdot 10^{-7}$ | $1.6178 \cdot 10^{-6}$ | 15 | $2.7884 \cdot 10^{-2}$ | $4.1507 \cdot 10^{-2}$ |
| 10 | $9.0788 \cdot 10^{-7}$ | $1.9066 \cdot 10^{-6}$ | $5.3927 \cdot 10^{-6}$ | 20 | $4.2648 \cdot 10^{-2}$ | $5.5343 \cdot 10^{-2}$ |

**Table 1**. Theoretical ranging accuracy for an AWGN channel with attenuation, $\sigma_D = \sqrt{\mathrm{CRB}(D)}$

$$\mathrm{HCRB}(D) = \frac{c^2 D^{\gamma_0+2}}{2\widetilde{k_0}^2 \frac{\mathcal{E}}{N_0}\left\{\left(M_2 - M_1^2 + \frac{M_1^2}{1+\frac{2\widetilde{k_0}^2}{X_c D_0^{\gamma}}\frac{\mathcal{E}}{N_0}}\right)\left(\frac{\gamma_0^2 c^2}{4} + \gamma_0 c D\overline{F'^2}\right) + M_2 D^2 \overline{F^2} - M_1^2\left(1 - \frac{1}{1+\frac{2\widetilde{k_0}^2}{X_c D_0^{\gamma}}\frac{\mathcal{E}}{N_0}}\right) D^2 (\overline{F'^2})^2\right\}}, \quad (13)$$

It is a loose bound to cope with nuisance parameters, when it is difficult to get the true CRB. The MCRB($D$) (14) depends on the average received energy. Taking $\widetilde{k_0}^2 = k_0^2/E[X^2]$ into (14), we find it is equal to the CRB($D$) (4). Due to the normalization, the average received energy in an AWGN channel with attenuation and shadowing is the same as the received energy in an AWGN channel only with attenuation.

The third bound is the Miller-Chang bound (MCB) proposed in [10]: $\mathrm{MCB}(D) = E_X\left\{1/[\mathbf{I}_D(\boldsymbol{\theta})]_{11}\right\}$, which is

$$\mathrm{MCB}(D) = E_X\left[\frac{1}{X^2}\right]\frac{c^2 D^{\gamma_0+2}}{2\widetilde{k_0}^2 \frac{\mathcal{E}}{N_0}\left(\frac{\gamma_0^2 c^2}{4} + D^2\overline{F^2} + \gamma_0 c D\overline{F'^2}\right)}. \quad (15)$$

It covers the estimator that is locally unbiased for all the values of the nuisance parameter $X$. This kind of estimator may not be achieved, since for extremely low signal strengths, the receiver can not detect the signal any more [10]. The term $1/[\mathbf{I}_D(\boldsymbol{\theta})]_{11}$ in MCB($D$) is the bound for the estimator with perfect knowledge of $X$.

The fourth bound is the extended MCB (EMCB) [8]: EMCB($D$) $= E_X\left\{\left[\mathbf{I}_D^{-1}(\boldsymbol{\theta})\right]_{11}\right\}$, which is

$$\mathrm{EMCB}(D) = E_X\left[\frac{1}{X^2}\right]\frac{c^2 D^{\gamma_0}}{2\widetilde{k_0}^2 \frac{\mathcal{E}}{N_0}\left\{\overline{F^2} - (\overline{F'^2})^2\right\}}. \quad (16)$$

It is the average over $X$ of the joint estimation bound, which assumes $X$ is an unknown deterministic parameter.

All the above bounds are related to each other as follows

$$\mathrm{CRB}(D) \geq \mathrm{HCRB}(D) \geq \mathrm{MCRB}(D), \quad (17)$$
$$\mathrm{EMCB}(D) \geq \mathrm{MCB}(D) \geq \mathrm{MCRB}(D). \quad (18)$$

The order (17) has already been proved in [11][12]. The first inequality indicates that the CRB($D$) (the true CRB for an AWGN channel with attenuation and shadowing) applied directly to $D$ using a marginal probability density function is tighter than the HCRB($D$). There is no performance improvement when estimating more parameters in the given system. When we have $\overline{F'^2} = 0$, for UWB signals, in the relevant ranges, we would observe HCRB($D$) $\approx$ MCRB($D$), since $\gamma_0^2 c^2/4 \ll D^2\overline{F^2}$. For NB signals, we expect to see that the MCRB($D$) is looser than the HCRB($D$). The order (18) is also verified in [8]. The inequality $[\mathbf{I}_D^{-1}(\boldsymbol{\theta})]_{11} \geq 1/[\mathbf{I}_D(\boldsymbol{\theta})]_{11}$ always holds, which confirms the inequality EMCB($D$) $\geq$ MCB($D$). Again relying on $\overline{F'^2} = 0$, for UWB signals, we would have MCB($D$) $\approx$ EMCB($D$). For NB signals, differences will be obvious.

Table 2 collects the ranging accuracy obtained by different bounds for UWB signals and NB signals. The parameters are set the same values as in the last section. The upper part of Table 2 is for UWB signals, while the lower part is for NB signals. The MCRB($D$) is independent of the shadowing effects, due to the channel energy normalization. It is equal to the CRB in an AWGN channel only with attenuation. The order EMCB($D$) $\geq$ MCB($D$) $\geq$ HCRB($D$) $\geq$ MCRB($D$) is established from Table 2 for UWB signals in the relevant ranges, which indicates that the prior knowledge of $X$ helps range estimation. As the shadowing effects increase, the estimators perform worse regardless of the prior knowledge of shadowing. For NB signals, the differences between the HCRB($D$) and the MCRB($D$), as well as between the MCB($D$) and the EMCB($D$) are obvious for the relevant ranges. The order EMCB($D$) $>$ HCRB($D$) still holds. The HCRB($D$) benefits from the prior knowledge of $X$. When $\sigma_x = 3$dB, the HCRB($D$) is larger than the MCB($D$) around the critical distance. However, this relationship does not retain, when $\sigma_x = 6$dB. As the shadowing effects become more serious, the MCB($D$) becomes larger even if we know the exact value of $X$ as assumed by the bound MCB($D$). This is due to the fact that it bounds the average performance of a kind of estimator, which relies on the instantaneous received energy. Its performance is unfavorable, when the instantaneous received energy is low. On the other hand, the HCRB($D$) is smaller when the shadowing is more severe under the condition that the channel energy is normalized. The prior knowledge of $X$ helps out when the instantaneous received energy is very low. In summary, it is important to take the prior knowledge into account to handle the shadowing effects.

Now let us investigate some estimators for $X$ and $D$. There are two different ways to estimate $X$ and $D$ depending on whether to employ the prior knowledge of $X$ in the estimation procedure. Method 1: If both $X$ and $D$ are assumed as unknown deterministic parameters, then the classic MLEs are derived. Method 2: if $X$ is assumed as a random variable and $p(X)$ is known, $D$ is assumed as an unknown deterministic parameter, then the Bayesian estimator is employed.

Method 1: we would like to find the $X$ and the $D$ that maximize $p(\mathbf{r}|X;D)$,

$$\frac{\partial \ln p(\mathbf{r}|X;D)}{\partial X} = 0, \quad (19)$$
$$\frac{\partial \ln p(\mathbf{r}|X;D)}{\partial D} = 0. \quad (20)$$

Solving (19), we obtain $\hat{X} = \frac{D^{\frac{\gamma_0}{2}}\mathcal{E}_{rs}(D)}{\widetilde{k_0}\mathcal{E}}$. Inserting $\hat{X}$ into (20)

| | $\sigma_{HCRB}$(m) | $\sigma_{MCRB}$(m) | $\sigma_{MCB}$(m) | $\sigma_{EMCB}$(m) | $\sigma_{HCRB}$(m) | $\sigma_{MCRB}$(m) | $\sigma_{MCB}$(m) | $\sigma_{EMCB}$(m) |
|---|---|---|---|---|---|---|---|---|
| $D$(m) | Shadowing $\sigma_x = 3$dB, UWB signals, $3.1$GHz $\sim 10.6$GHz | | | | Shadowing $\sigma_x = 6$dB, UWB signals, $3.1$GHz $\sim 10.6$GHz | | | |
| 1 | $9.0788 \cdot 10^{-8}$ | $9.0786 \cdot 10^{-8}$ | $1.0089 \cdot 10^{-7}$ | $1.0090 \cdot 10^{-7}$ | $9.0788 \cdot 10^{-8}$ | $9.0786 \cdot 10^{-8}$ | $1.3822 \cdot 10^{-7}$ | $1.3822 \cdot 10^{-7}$ |
| 3 | $2.7236 \cdot 10^{-7}$ | $2.7236 \cdot 10^{-7}$ | $3.0269 \cdot 10^{-7}$ | $3.0269 \cdot 10^{-7}$ | $2.7236 \cdot 10^{-7}$ | $2.7236 \cdot 10^{-7}$ | $4.1466 \cdot 10^{-7}$ | $4.1466 \cdot 10^{-7}$ |
| 10 | $9.0788 \cdot 10^{-7}$ | $9.0788 \cdot 10^{-7}$ | $1.0090 \cdot 10^{-6}$ | $1.0090 \cdot 10^{-6}$ | $9.0788 \cdot 10^{-7}$ | $9.0788 \cdot 10^{-7}$ | $1.3822 \cdot 10^{-6}$ | $1.3822 \cdot 10^{-6}$ |
| $D$(m) | Shadowing $\sigma_x = 3$dB, NB signals, 0MHz $\sim$ 5MHz | | | | Shadowing $\sigma_x = 6$dB, NB signals, 0MHz $\sim$ 5MHz | | | |
| 10 | $2.5908 \cdot 10^{-2}$ | $1.4317 \cdot 10^{-2}$ | $1.5910 \cdot 10^{-2}$ | $3.0750 \cdot 10^{-2}$ | $2.2456 \cdot 10^{-2}$ | $1.4317 \cdot 10^{-2}$ | $2.1793 \cdot 10^{-2}$ | $4.2121 \cdot 10^{-2}$ |
| 15 | $4.0265 \cdot 10^{-2}$ | $2.7884 \cdot 10^{-2}$ | $3.0986 \cdot 10^{-2}$ | $4.6125 \cdot 10^{-2}$ | $3.7419 \cdot 10^{-2}$ | $2.7884 \cdot 10^{-2}$ | $4.2445 \cdot 10^{-2}$ | $6.3182 \cdot 10^{-2}$ |
| 20 | $5.4390 \cdot 10^{-2}$ | $4.2648 \cdot 10^{-2}$ | $4.7393 \cdot 10^{-2}$ | $6.1500 \cdot 10^{-2}$ | $5.2070 \cdot 10^{-2}$ | $4.2648 \cdot 10^{-2}$ | $6.4919 \cdot 10^{-2}$ | $8.4242 \cdot 10^{-2}$ |

**Table 2**. Theoretical ranging accuracy for an AWGN channel with attenuation and shadowing $\sigma = \sqrt{\text{CRB}}$

says that the estimation of $D$ should satisfy the following equation

$$\int_0^{T_o} r(t)\frac{\partial s(t-\tau)}{\partial \tau}dt + \overline{F'^2}\mathcal{E}_{rs}(D) = 0. \tag{21}$$

Define $\overline{{F'_{rs}}^2}(D) = \int_0^{T_o} r(t)\frac{\partial s(t-\tau)}{\partial\tau}dt \Big/ \mathcal{E}_{rs}(D)$ conditioned on $\mathcal{E}_{rs}(D) \neq 0$. Then the MLE of $D$ is equivalent to

$$\hat{D} = \underset{0<D<cT_o}{\text{argmin}} \left|\overline{{F'_{rs}}^2}(D)\right|. \tag{22}$$

In reality, we do not check the condition $\mathcal{E}_{rs}(D) \neq 0$, but test whether $\mathcal{E}_{rs}(D)$ is above the noise floor. If it is true, then we conclude that $\mathcal{E}_{rs}(D) \neq 0$ is satisfied. Otherwise, $\mathcal{E}_{rs}(D)$ equals zero and we have to check other $D$ candidates. The average performance of this estimator can asymptotically approach EMCB$(D)$.

Method 2: In this case, we have prior knowledge of $X$, and a Bayesian estimator can be employed. The well-known minimum mean square error (MMSE) estimator is first employed $\hat{X} = E(X|\mathbf{r};D) = \int Xp(X|\mathbf{r};D)dX$, where $p(X|\mathbf{r};D) = p(\mathbf{r}|X;D)p(X) \Big/ \int p(\mathbf{r}|X;D)p(X)dX$. The integration in the denominator prevents us from finding a closed form. Hence, we resort to a maximum a posteriori (MAP) estimator, which boils down to finding the $X$ that maximizes $p(X|\mathbf{r};D)$. It is equivalent to maximizing $p(\mathbf{r}|X;D)p(X)$ or ln $(p(\mathbf{r}|X;D)p(X))$. Then, the joint estimation of $X$ and $D$ is

$$\begin{aligned} [\hat{X}, \hat{D}] &= \underset{X,0<D<cT_o}{\text{argmax}} \{\ln(p(\mathbf{r}|X;D)p(X))\} \\ &= \underset{X,0<D<cT_o}{\text{argmin}} \{J(X,D)\}. \end{aligned} \tag{23}$$

where $J(X,D) = g_1(X,D) + g_2(X)$,

$$\begin{aligned} g_1(X,D) &= \frac{\mathcal{E}}{N_0}\left(\frac{\widetilde{k_0}X}{D^{\frac{\gamma_0}{2}}} - \frac{\mathcal{E}_{rs}(D)}{\mathcal{E}}\right)^2 - \frac{\mathcal{E}_{rs}^2(D)}{N_0\mathcal{E}}, \\ g_2(X) &= \frac{200}{\sigma_x^2 \ln^2 10}\left(\ln X + \frac{\sigma_x^2 \ln^2 10}{400}\right)^2 - \frac{\sigma_x^2 \ln^2 10}{800}. \end{aligned}$$

Method 2 is much more complicated than Method 1, since it has to execute a two-dimensional search compared to a one-dimensional search in Method 1. The performance limitation of Method 1 and Method 2 for range estimation can be indicated by the bound EMCB$(D)$ and the bound HCRB$(D)$, respectively.

## 4. CONCLUSIONS

In this paper, we have investigated the accuracy for range estimation by a new method, which exploits the range information in both the amplitude and the time delay of the received signal. Several bounds are derived not only in an AWGN channel with attenuation , but also in an AWGN channel with attenuation and shadowing. For UWB signals, the new method does not show obvious benefits. However, it indeed improves the ranging accuracy using NB signals. Moreover, taking the prior knowledge of the shadowing effects into account lowers the bounds for range estimation. Further work will focus on range estimation using UWB signals in multipath environments. Since the first path is the most relevant to the TOA estimation, we would like to explore the statistical properties of the first path and take full use of it in ranging.

## 5. REFERENCES


[1] S. Gezici, Z. Tian, G.B. Giannakis, H. Kobayashi, A.F. Molisch, H.V. Poor, and Z. Sahinoglu, "Localization via ultra-wideband radios: a look at positioning aspects for future sensor networks," *IEEE Signal Process. Mag.*, vol. 22, pp. 70–84, July 2005.

[2] J. Zhang, R. A. Kennedy, and T. D. Abhayapala, "Cramér-Rao lower bounds for the synchronization of UWB signals," *EURASIP J. Appl. Signal Proc.*, vol. 3, pp. 426–438, 2005.

[3] R. Cardinali, L. De Nardis, M.-G. Di Benedetto, and P. Lombardo, "UWB ranging accuracy in high- and low-data-rate applications," *IEEE Trans. Microw. Theory Tech.*, vol. 54, pp. 1865–1875, June 2006.

[4] Y. Qi and H. Kobayashi, "On relation among time delay and signal strength based geolocation methods," *IEEE GLOBECOM '03.*, vol. 7, pp. 4079–4083, Dec. 2003.

[5] A. Catovic and Z. Sahinoglu, "The Cramer-Rao bounds of hybrid TOA/RSS and TDOA/RSS location estimation schemes," *IEEE Commun. Lett.*, vol. 8, pp. 626–628, Oct. 2004.

[6] T.S. Rappaport, *Wireless communications principles and practice*, Prentice Hall, 1996.

[7] H.L. Van Trees, *Detection, estimation, and modulation theory*, John Wiley & Sons, Inc., New York, 1968.

[8] F. Gini and R. Reggiannini, "On the use of Cramer-Rao-like bounds in the presence of random nuisance parameters," *IEEE Trans. Commun.*, vol. 48, pp. 2120–2126, Dec 2000.

[9] A.N. D'Andrea, U. Mengali, and R. Reggiannini, "The modified Cramer-Rao bound and its application to synchronization problems," *IEEE Trans. Commun.*, vol. 42, no. 234, pp. 1391–1399, Feb/Mar/Apr 1994.

[10] R. Miller and Chow Chang, "A modified Cramér-Rao bound and its applications," *IEEE Trans. Inf. Theory*, vol. 24, pp. 398–400, May 1978.

[11] B. Z. Bobrovsky, E. Mayer-Wolf, and M. Zakai, "Some classes of global Cramer-Rao bounds," *The Annals of Statistics*, vol. 15, no. 4, pp. 1421–1438, Dec 1987.

[12] I. Reuven and H. Messer, "A Barankin-type lower bound on the estimation error of a hybrid parameter vector," *IEEE Trans. Inf. Theory*, vol. 43, pp. 1084–1093, May 1997.