Extending the Classical Multidimensional Scaling Algorithm Given Partial Pairwise Distance Measurements

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***Abstract*—We consider the problem of node localization given partial pairwise distance measurements. Current solutions first complete the missing distances and then apply the classical multidimensional scaling (MDS) algorithm. Instead, we extend the classical MDS to a setup where the sensor network is composed of a fully connected group of nodes that communicate with each other (e.g., beacons), and a group of nodes that cannot communicate with each other, but each one of them communicates with each node in the first group. The positions of all nodes are unknown. We localize the fully connected nodes by exploiting their distance measurements to the disconnected nodes. At the same time, the positions of the disconnected nodes are obtained up to a translation relative to the positions of the connected nodes. Recovering this translation, can be obtained with an additional step. Simulation results show that the proposed algorithm outperforms current MDS-like solutions to the problem.**

***Index Terms*—Multidimensional scaling, node localization, partial network connectivity, wireless sensor networks.**

I. Introduction

NODE localization from pairwise Euclidean distance measurements has become a fundamental research topic with the growing interest in wireless sensor networks [1]. The classical MDS algorithm [2] transforms the problem into an eigenvalue problem of a so-called projected distance matrix which solely depends on the distances. This matrix has rank two (for a planar geometry) and its first two eigenvectors provide the coordinates of all nodes up to a translation, rotation, and reflection [3]–[9]. Applying the MDS algorithm requires a fully connected sensor network, i.e., given a sensor network with N nodes, we need $N(N-1)/2$ distance measurements.

However, in practice, only a limited number of distance measurements are given due to communication limits imposed on the nodes (e.g., battery constraints). For simplicity, assume we have an ad-hoc sensor network with N nodes where the positions of all the nodes are unknown. Only M nodes (in practice, $M \ll N$) in the network can communicate with all the other nodes. Such nodes can be beacon nodes which usually have less stringent communication constraints. The other $N-M$ nodes cannot communicate with each other due to communication limits, but communicate with the first M nodes. One possible solution to this setup is the scaling by majorizing a complicated function (SMACOF) algorithm, which is based on iteratively minimizing a global stress cost function [3], [11] composed of: a weighted least squares (LS) part and possibly a penalty term that includes prior information about node positions. This approach involves a highly nonlinear cost function which requires many initial guesses of the node positions to ensure convergence to the global minimum. As a result, suboptimal solutions have been investigated, which can possibly be used as an initial point of this algorithm. Examples are MDS-MAP [8], and SVD-Reconstruct [9], which both are based on first completing the missing entries in the distance matrix and then applying the classical MDS, or Nyström's algorithm [10] which is used to reduce the complexity of the singular value decomposition step involved in the classical MDS algorithm. With Nyström's algorithm the positions of the first group are estimated using the classical MDS method, while the positions of the second group are estimated using the LS method based on the former results and the mutual measurements between the two groups [10, eqs. (8)–(9)].

Herein, we suggest a localization approach which extends the classical MDS to the current setup. We localize the nodes of the first group by exploiting their distance measurements to the disconnected nodes. At the same time, the second group's positions are obtained up to a translation relative to the former nodes. If recovering this translation is also of interest, then a possible additional step can be implemented. We examine our results with Monte-Carlo simulations by evaluating the normalized root mean square error (RMSE) between the true inter-node distances and their estimates. We compare our two-step approach with the MDS-MAP [8], the SVD-Reconstruct [9], and Nyström's algorithm [10]. Simulation results show that our proposed approach outperforms these previous solutions.

II. Problem Formulation

Consider N sensor nodes randomly distributed in a two-dimensional plane. Let $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N] \in \mathcal{R}^{2\times N}$ consist of all the node positions, where $\mathbf{x}_n \in \mathcal{R}^{2\times 1}$, $n = 1, 2, \ldots, N$, is the position of the nth node. We assume that we have two groups of nodes: 1) M nodes which are fully connected; 2) $N-M$ nodes which are fully disconnected (i.e., each of the nodes in the second group does not communicate with the other nodes in this group, but communicates with each node in the first group.) Let $\mathbf{X}_1 = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_M] \in \mathcal{R}^{2\times M}$ and $\mathbf{X}_2 = [\mathbf{x}_{M+1}, \cdots, \mathbf{x}_N] \in \mathcal{R}^{2\times(N-M)}$ represent the set of node positions of the first group and the second group, respectively. The distance measurements between the ith and jth nodes (given

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they are connected) is $r_{i,j} = d_{i,j} + e_{i,j}$, where $d_{i,j} = ||\mathbf{x}_i - \mathbf{x}_j||$, and $e_{i,j} \sim \mathcal{N}(0, \sigma^2)$ is the uncorrelated additive noise, where σ^2 is a known noise variance. The problem discussed herein is briefly stated as follows: Given the available pairwise measurements $\{r_{i,j}\}$, determine the positions of the nodes in an arbitrary reference system (up to rotation, reflection and translation, where absolute orientation is obtained using reference points). In other words, our goal is to reconstruct the constellation of the sensor network given partial pairwise node measurements.

III. Possible Position Projections

Let us first consider the noiseless case. The available squared distance measurements can then be written in a matrix form as

$$\begin{aligned}(\mathbf{R}\odot\mathbf{R})\odot\mathbf{W} &= \left(\boldsymbol{\psi}\mathbf{1}_N^T - 2\mathbf{X}^T\mathbf{X} + \mathbf{1}_N\boldsymbol{\psi}^T\right)\odot\mathbf{W} \\ &= \mathbf{W}diag(\boldsymbol{\psi}) - 2\mathbf{X}^T\mathbf{X}\odot\mathbf{W} + diag(\boldsymbol{\psi})\mathbf{W} \quad (1)\end{aligned}$$

where $\boldsymbol{\psi} = [||\mathbf{x}_1||^2, \ldots, ||\mathbf{x}_N||^2]^T$, $\odot$ is the Hadamard product, $\mathbf{1}_n \in \mathcal{R}^{n\times 1}$ is a vector with all elements equal to one, and $\mathbf{W} \in \mathcal{R}^{N\times N}$ is the symmetric communication connectivity matrix of the network, with its (i,j)th element, denoted by $w_{i,j}$, equal to one if the ith node and the jth node communicate with each other, and equal to zero if they do not communicate with each other. The elements on the diagonal of this matrix are arbitrary. The positions of the nodes appear in the second term. The idea is therefore to omit the first and the last terms in (1) while keeping the second term. We omit these two terms by pre- and post-multiplying (1) by an orthogonal projection matrix $\mathbf{P}_w \in \mathcal{R}^{N\times N}$ such that $\mathbf{P}_w\mathbf{W} = \mathbf{W}\mathbf{P}_w = \mathbf{0}$. This operation can be considered as an extension of the classical MDS. We assume that the structure of $\mathbf{W}$ is

$$\mathbf{W} = \begin{bmatrix} \mathbf{1}_M\mathbf{1}_M^T & \mathbf{1}_M\mathbf{1}_{N-M}^T \\ \mathbf{1}_{N-M}\mathbf{1}_M^T & \boldsymbol{\Lambda} \end{bmatrix} \quad (2)$$

where $\boldsymbol{\Lambda} = diag(\lambda_1, \lambda_1, \ldots, \lambda_{N-M})$. Given $\{\lambda_j\}_{j=1}^{N-M}$ we determine $\mathbf{P}_w$. Let $\tilde{\mathbf{W}}$ be the matrix containing a basis for the column span of $\mathbf{W}$. Then $\mathbf{P}_w = \mathbf{I}_N - \tilde{\mathbf{W}}(\tilde{\mathbf{W}}^T\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{W}}^T$. We discuss two possible types of $\tilde{\mathbf{W}}$, and present the effect of each projection on the node positions. We emphasize that in the proposed localization approach these projections are not directly applied to the positions. Still, the purpose is to give an intuition for choosing these projections by presenting their effect on the positions of the nodes.

A. Projection Type A

Assume $\boldsymbol{\Lambda} = \mathbf{0}_{N-M}\mathbf{0}_{N-M}^T$, where $\mathbf{0}_n \in \mathcal{R}^{n\times 1}$ is a vector with all elements equal to zero. Then $\tilde{\mathbf{W}}^{(A)}$ and its associated orthogonal projection matrix $\mathbf{P}_w^{(A)}$ are

$$\tilde{\mathbf{W}}^{(A)} = \begin{bmatrix} \mathbf{1}_M & \mathbf{1}_M \\ \mathbf{1}_{N-M} & \mathbf{0}_{N-M} \end{bmatrix} \in \mathcal{R}^{N\times 2} \quad (3)$$

$$\mathbf{P}_w^{(A)} = \begin{bmatrix} \mathbf{I}_M - \frac{1}{M}\mathbf{1}_M\mathbf{1}_M^T & \mathbf{0}_M\mathbf{0}_{N-M}^T \\ \mathbf{0}_{N-M}\mathbf{0}_M^T & \mathbf{I}_{N-M} - \frac{1}{N-M}\mathbf{1}_{N-M}\mathbf{1}_{N-M}^T \end{bmatrix}. \quad (4)$$

The projected node positions are $\mathbf{P}_w^{(A)}\mathbf{X}^T = [\tilde{\mathbf{X}}_1\ \tilde{\mathbf{X}}_2]^T$ where

$$\tilde{\mathbf{X}}_1 = \mathbf{X}_1 - \mathbf{x}_1^{(c)}\mathbf{1}_M^T \quad (5)$$

$$\tilde{\mathbf{X}}_2 = \mathbf{X}_2 - \mathbf{x}_2^{(c)}\mathbf{1}_{N-M}^T \quad (6)$$

with $\mathbf{x}_1^{(c)} = (1/M)\sum_{j=1}^{M}\mathbf{x}_j$ and $\mathbf{x}_2^{(c)} = (1/N - M)\sum_{j=M+1}^{N}\mathbf{x}_j$ the centers of gravity of the first node group and second node group, respectively. The advantage of this projection is that it decouples the two groups, but the disadvantage is that both groups are translated to the origin. To obtain the relative distance between the centers of gravity of the two groups we need the following projection.

B. Projection Type B

Assume $\boldsymbol{\Lambda} = (N-M)\mathbf{I}_{N-M}$. Then $\tilde{\mathbf{W}}^{(B)}$ and its associated orthogonal projection matrix $\mathbf{P}_w^{(B)}$ are

$$\tilde{\mathbf{W}}^{(B)} = \begin{bmatrix} \mathbf{1}_M\mathbf{1}_{N-M}^T \\ \boldsymbol{\Lambda} \end{bmatrix} \in \mathcal{R}^{N\times(N-M)} \quad (7)$$

$$\mathbf{P}_w^{(B)} = \begin{bmatrix} \mathbf{I}_M - \frac{1}{N}\mathbf{1}_M\mathbf{1}_M^T & -\frac{1}{N}\mathbf{1}_M\mathbf{1}_{N-M}^T \\ -\frac{1}{N}\mathbf{1}_{N-M}\mathbf{1}_M^T & \frac{M}{(N-M)N}\mathbf{1}_{N-M}\mathbf{1}_{N-M}^T \end{bmatrix}. \quad (8)$$

The projected node positions are

$$\mathbf{P}_w^{(B)}\mathbf{X}^T = \begin{bmatrix} \mathbf{I}_M & \mathbf{q}_1 \\ \mathbf{0}_{N-M}\mathbf{0}_M^T & -\mathbf{q}_2 \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{X}}_1^T \\ \mathbf{u}^T \end{bmatrix} \quad (9)$$

where the relative distance vector between the centers of gravity of the two groups is defined as

$$\mathbf{u} = \mathbf{x}_1^{(c)} - \mathbf{x}_2^{(c)} \quad (10)$$

and $\mathbf{q}_1 = (N - M/N)\mathbf{1}_M$, $\mathbf{q}_2 = (M/N)\mathbf{1}_{N-M}$. The advantage of using this projection is that it retains the relative translation between the centers of gravity of the two groups, while the disadvantage is that all the nodes of the second group are translated to the same position.

IV. The Proposed Localization Approach

The idea of reconstructing the configuration of the sensor network is as follows: use $\mathbf{P}_w^{(A)}$ to estimate $\tilde{\mathbf{X}}_1$, and $\tilde{\mathbf{X}}_2$. If one is also interested in the relative translation vector $\mathbf{u}$, then first use $\mathbf{P}_w^{(B)}$ to estimate $\mathbf{u}$ (given the estimate of $\tilde{\mathbf{X}}_1$), and then update the estimate of $\tilde{\mathbf{X}}_2$ by $\tilde{\mathbf{X}}_2' = \tilde{\mathbf{X}}_2 + \mathbf{u}$. We now discuss each of these steps in detail.

A. The Result of Using Projection $\mathbf{P}_w^{(A)}$

By post- and pre-multiplying (1) by $(1/\sqrt{2})\mathbf{P}_w^{(A)}$ we get

$$\begin{aligned}\mathbf{B} \triangleq \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{12}^T & \mathbf{B}_{22} \end{bmatrix} &= -\frac{1}{2}\mathbf{P}_w^{(A)}\left((\mathbf{R}\odot\mathbf{R})\odot\mathbf{W}\right)\mathbf{P}_w^{(A)} \\ &= \mathbf{P}_w^{(A)}(\mathbf{X}^T\mathbf{X}\odot\mathbf{W})\mathbf{P}_w^{(A)} \\ &= \begin{bmatrix} \tilde{\mathbf{X}}_1^T\tilde{\mathbf{X}}_1 & \tilde{\mathbf{X}}_1^T\tilde{\mathbf{X}}_2 \\ \tilde{\mathbf{X}}_2^T\tilde{\mathbf{X}}_1 & \mathbf{0}_{N-M}\mathbf{0}_{N-M}^T \end{bmatrix}. \quad (11)\end{aligned}$$

We recover $\tilde{\mathbf{X}}_1, \tilde{\mathbf{X}}_2$ (up to translation, reflection or rotation) as $[\hat{\tilde{\mathbf{X}}}_1, \hat{\tilde{\mathbf{X}}}_2] = \arg\min_{[\tilde{\mathbf{X}}_1, \tilde{\mathbf{X}}_2]} J(\tilde{\mathbf{X}}_1, \tilde{\mathbf{X}}_2)$ where

$$\begin{aligned}J(\tilde{\mathbf{X}}_1, \tilde{\mathbf{X}}_2) &= \left\|\mathbf{B} - \mathbf{P}_w^{(A)}(\mathbf{X}^T\mathbf{X}\odot\mathbf{W})\mathbf{P}_w^{(A)}\right\|_F^2 \\ &= \left\|\mathbf{B}_{11} - \tilde{\mathbf{X}}_1^T\tilde{\mathbf{X}}_1\right\|_F^2 + 2\left\|\mathbf{B}_{12} - \tilde{\mathbf{X}}_1^T\tilde{\mathbf{X}}_2\right\|_F^2. \quad (12)\end{aligned}$$

It is clear that $\tilde{\mathbf{X}}_2$ which minimizes (12) is

$$\hat{\tilde{\mathbf{X}}}_2 = \left(\tilde{\mathbf{X}}_1\tilde{\mathbf{X}}_1^T\right)^{-1}\tilde{\mathbf{X}}_1\mathbf{B}_{12}. \quad (13)$$

Substituting (13) into (12) yields

$$\begin{aligned} J_1(\tilde{\mathbf{X}}_1) &= \left\|\mathbf{B}_{11} - \tilde{\mathbf{X}}_1^T\tilde{\mathbf{X}}_1\right\|_F^2 \\ &\quad + 2\left\|\mathbf{B}_{12} - \tilde{\mathbf{X}}_1^T\left(\tilde{\mathbf{X}}_1\tilde{\mathbf{X}}_1^T\right)^{-1}\tilde{\mathbf{X}}_1\mathbf{B}_{12}\right\|_F^2 \\ &= \mathrm{tr}\left(\mathbf{B}_{11}^T\mathbf{B}_{11} + 2\mathbf{B}_{12}^T\mathbf{B}_{12}\right. \\ &\quad - 2\mathbf{B}_{11}\tilde{\mathbf{X}}_1^T\tilde{\mathbf{X}}_1 + \tilde{\mathbf{X}}_1^T\tilde{\mathbf{X}}_1\tilde{\mathbf{X}}_1^T\tilde{\mathbf{X}}_1 \\ &\quad \left. -2\mathbf{B}_{12}^T\tilde{\mathbf{X}}_1^T\left(\tilde{\mathbf{X}}_1\tilde{\mathbf{X}}_1^T\right)^{-1}\tilde{\mathbf{X}}_1\mathbf{B}_{12}\right). \quad (14) \end{aligned}$$

By taking the derivative with respect to (w.r.t.) $\tilde{\mathbf{X}}_1^T$ we obtain

$$\begin{aligned} &\frac{\partial J_1(\tilde{\mathbf{X}}_1)}{\partial\tilde{\mathbf{X}}_1^T} = -\mathbf{B}_{11}\tilde{\mathbf{X}}_1^T + \tilde{\mathbf{X}}_1^T\tilde{\mathbf{X}}_1\tilde{\mathbf{X}}_1^T + \tilde{\mathbf{X}}_1^T\left(\tilde{\mathbf{X}}_1\tilde{\mathbf{X}}_1^T\right)^{-1} \\ &\times\tilde{\mathbf{X}}_1\mathbf{B}_{12}\mathbf{B}_{12}^T\tilde{\mathbf{X}}_1^T\left(\tilde{\mathbf{X}}_1\tilde{\mathbf{X}}_1^T\right)^{-1} - \mathbf{B}_{12}\mathbf{B}_{12}^T\tilde{\mathbf{X}}_1^T\left(\tilde{\mathbf{X}}_1\tilde{\mathbf{X}}_1^T\right)^{-1}. \quad (15) \end{aligned}$$

Since no closed-form expression for $\tilde{\mathbf{X}}_1$ that zeros (15) exists, we estimate it as detailed in Algorithm 1. The final estimate is denoted by $\hat{\tilde{\mathbf{X}}}_1$. After obtaining $\hat{\tilde{\mathbf{X}}}_1$, we substitute the result in (13) and obtain $\hat{\tilde{\mathbf{X}}}_2$.

Algorithm 1 Estimating the positions of the fully connected nodes

- Initial step: $\hat{\tilde{\mathbf{X}}}_1^{(MDS)} = \mathbf{\Phi}^{1/2}\mathbf{C}^T$, where $\mathbf{\Phi} \in \mathcal{R}^{2\times 2}$, $\mathbf{C} \in \mathcal{R}^{M\times 2}$ contain the two largest eigenvalues of $\mathbf{B}_{11}$, and their two associated orthonormal eigenvectors, respectively.
- Updating step: Given $\hat{\tilde{\mathbf{X}}}_1^{(MDS)}$ determine $\hat{\tilde{\mathbf{X}}}_1$ as follows.
  1) Let $\hat{\tilde{\mathbf{X}}}_1^{(k)}$ be the estimate of $\tilde{\mathbf{X}}_1$ at the kth iteration step.
  2) Calculate the gradient $\mathbf{G}^{(k)} = (\partial J_1(\tilde{\mathbf{X}}_1)/\partial\tilde{\mathbf{X}}_1^T)|_{\tilde{\mathbf{X}}_1=\hat{\tilde{\mathbf{X}}}_1^{(k)}}$ according to (15).
  3) Update the step size, $\mu^{(k)}$, as $\mu^{(k)} = \underset{\mu}{\mathrm{argmin}}\, J_1(\hat{\tilde{\mathbf{X}}}_1^{(k)} - \mu\mathbf{G}^{(k)})$ using (14).
  4) Update the estimate, $\hat{\tilde{\mathbf{X}}}_1^{(k+1)} = \hat{\tilde{\mathbf{X}}}_1^{(k)} - \mu^{(k)}\mathbf{G}^{(k)}$.
  5) if $|J_1(\hat{\tilde{\mathbf{X}}}_1^{(k+1)}) - J_1(\hat{\tilde{\mathbf{X}}}_1^{(k)})| < \epsilon$ where ϵ is a predefined tolerance, then $\hat{\tilde{\mathbf{X}}}_1 = \hat{\tilde{\mathbf{X}}}_1^{(k+1)}$, else perform steps 1–4.

B. The Result of Using Projection $\mathbf{P}_w^{(B)}$

Given the estimated positions of the previous step, we are interested in estimating the relative distance vector between the two groups in order to reconstruct the complete network configuration. By post- and pre-multiplying (1) by $(1/\sqrt{2})\mathbf{P}_w^{(B)}$, we get after a few mathematical steps that

$$\begin{aligned} \mathbf{F} &\triangleq \begin{bmatrix}\mathbf{F}_{11} & \mathbf{F}_{12}\\ \mathbf{F}_{12}^T & \mathbf{F}_{22}\end{bmatrix} = -\frac{1}{2}\mathbf{P}_w^{(B)}\left((\mathbf{R}\odot\mathbf{R})\odot\mathbf{W}\right)\mathbf{P}_w^{(B)} \\ &= \mathbf{P}_w^{(B)}(\mathbf{X}^T\mathbf{X}\odot\mathbf{W})\mathbf{P}_w^{(B)} \\ &= \begin{bmatrix}\tilde{\mathbf{X}}_1^T\tilde{\mathbf{X}}_1 + \mathbf{q}_1\mathbf{u}^T\tilde{\mathbf{X}}_1 + \tilde{\mathbf{X}}_1^T\mathbf{u}\mathbf{q}_1^T & -\tilde{\mathbf{X}}_1^T\mathbf{u}\mathbf{q}_2^T \\ -\mathbf{q}_2\mathbf{u}^T\tilde{\mathbf{X}}_1 & \mathbf{0}_{N-M}\mathbf{0}_{N-M}^T\end{bmatrix} \\ &\quad + \gamma\mathbf{q}\mathbf{q}^T \quad (16) \end{aligned}$$

where $\mathbf{q} = [\mathbf{q}_1^T, -\mathbf{q}_2^T]^T$, $\gamma = \|\mathbf{u}\|^2 + \eta^2$, and $\eta^2 = (1/N - M)\sum_{j=m+1}^{N}\|\mathbf{x}_j\|^2 - \|\mathbf{x}_2^{(c)}\|^2$. Note that η^2 describes the scattering radius around the center of gravity of the positions of the nodes in the second group. We are interested in estimating the relative distance $\mathbf{u}$ given the translated positions of the first group $\tilde{\mathbf{X}}_1$, and consider γ as a nuisance parameter. To estimate $\mathbf{u}$ we minimize

$$\begin{aligned} H(\mathbf{u},\gamma) &= \left\|\mathbf{F}_{11} - \left(\tilde{\mathbf{X}}_1^T\tilde{\mathbf{X}}_1 + \mathbf{q}_1\mathbf{u}^T\tilde{\mathbf{X}}_1 + \tilde{\mathbf{X}}_1^T\mathbf{u}\mathbf{q}_1^T + \gamma\mathbf{q}_1\mathbf{q}_1^T\right)\right\|_F^2 \\ &\quad + 2\left\|\mathbf{F}_{12} - \left(\tilde{\mathbf{X}}_1^T\mathbf{u}\mathbf{q}_2^T - \gamma\mathbf{q}_1\mathbf{q}_2^T\right)\right\|_F^2 + \left\|\mathbf{F}_{22} - \gamma\mathbf{q}_2\mathbf{q}_2^T\right\|_F^2. \quad (17) \end{aligned}$$

By taking the derivative w.r.t. $\mathbf{u}$ and equating the result to zero, and using the fact that $(N - M/N)\mathbf{1}_M^T\tilde{\mathbf{X}}_1^T\mathbf{u} = 0$ (since the center of gravity of $\tilde{\mathbf{X}}_1$ is at the origin), we get that the estimated relative distance vector is

$$\hat{\mathbf{u}} = \left(\tilde{\mathbf{X}}_1\tilde{\mathbf{X}}_1^T\right)^{-1}\tilde{\mathbf{X}}_1\left(\frac{1}{M}\mathbf{F}_{11}\mathbf{1}_M - \frac{1}{N-M}\mathbf{F}_{12}\mathbf{1}_{N-M}\right). \quad (18)$$

Note that $\hat{\mathbf{u}}$ does not depend on the estimate of the nuisance parameter γ, and we therefore do not proceed in estimating it. We now substitute $\hat{\tilde{\mathbf{X}}}_1$ (obtained in the first step) instead of $\tilde{\mathbf{X}}_1$ in (18). We then update $\hat{\tilde{\mathbf{X}}}_2$ (obtained in the first step) by $\hat{\tilde{\mathbf{X}}}_2' = \hat{\tilde{\mathbf{X}}}_2 + \hat{\mathbf{u}}\mathbf{1}_{N-M}^T$. Finally, $\hat{\mathbf{X}} = [\hat{\tilde{\mathbf{X}}}_1\ \hat{\tilde{\mathbf{X}}}_2'] \in \mathcal{R}^{2\times N}$ contains the estimated node positions up to rotation, reflection, and translation w.r.t. the original configuration (which can be corrected using anchors in the network).

V. Numerical Results

We compare our proposed algorithm with the MDS-MAP method [8], the SVD-Reconstruct method [9] (where each missing entry is replaced by zero since it is the optimal choice as indicated in [9]), and with Nyström's method [10]. We also show some results for the SMACOF algorithm [11, pp. 150–157], one for a random initial point and one for an improved initial point (our proposed method). We consider a square area of 100×100 $[m \times m]$, and $N = 100$ nodes randomly positioned. The tolerance of the iterative search in the first estimation step is $\epsilon = 0.005$.

In the first simulation, we consider the cases of $M = 10$, $M = 50$ and $M = 90$. For each case we varied the noise variance from $\sigma^2 = 0.2$ to $\sigma^2 = 5$. For each noise variance we consider $K = 50$ configuration realizations, and for each configuration we perform $N_{exp} = 50$ Monte-Carlo (MC) trials. Since all positions are unknown, we define the RMSE as $\overline{\varepsilon} = \sqrt{(1/(KN_{exp}))\sum_{k=1}^{K\cdot N_{exp}}\overline{\varepsilon}_k^2[m]}$, where $\overline{\varepsilon}_k = \sqrt{(1/N_d)\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}w_{i,j}(\hat{d}_{i,j} - d_{i,j})^2[m]}$, with $\hat{d}_{i,j}$ the estimated distance between the connected nodes i and j at one configuration and one MC trial, and N_d is the number of connections (in this case $N_d = (1/2)M(N-1)$). The results are shown in Fig. 1. As can be seen, our proposed algorithm has the smallest RMSE, compared to other MDS-like solutions. Both MDS-MAP and SVD-Reconstruct have worse performance when the number of connections is

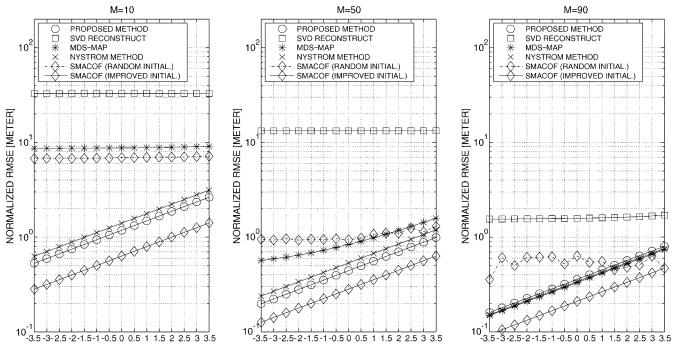

Fig. 1. (Left) RMSEs versus the noise variance for $M = 10$, (center) $M = 50$, and (right) $M = 90$.

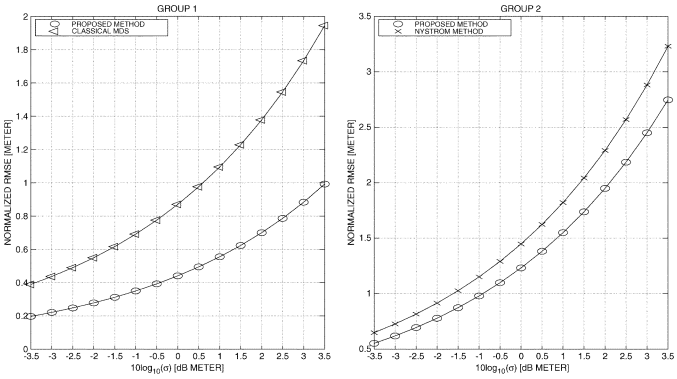

Fig. 2. (Left) RMSEs of the initial solution and the iterative solution of the first group and (right) the RMSE of the second group using the proposed method and Nyström's method.

small $(M = 10)$, however, when the number of the connections is large $(M = 90)$, MDS-MAP has a similar performance as the proposed algorithm and the Nyström's method. From the results, we observe that a good initial point for the SMACOF algorithm is crucial and it is clear that the better the initial point, the faster the convergence of SMACOF. That is why our method is preferred over other MDS-like solutions if an initial point for SMACOF is sought for. In Fig. 2 (left plot) we compare the RMSE of $\hat{\hat{\mathbf{X}}}_1$ with that of the initial estimate $\hat{\hat{\mathbf{X}}}_1^{(MDS)}$ (in this case $N_d = M(M-1)/2$), and in Fig. 2 (right plot) we compare the RMSE of $\hat{\hat{\mathbf{X}}}_2'$ of the proposed method with that of Nyström's method (in this case $N_d = (1/2)M(N-M)$). For both cases we assume that $M = 10$. As can be seen, when the noise variance increases, the gap between the RMSEs of $\hat{\hat{\mathbf{X}}}_1$ and $\hat{\hat{\mathbf{X}}}_1^{(MDS)}$ increases, and thus the iterative solution improves the RMSE of the initial estimate. The RMSE of $\hat{\hat{\mathbf{X}}}_2'$ is also improved w.r.t. the Nyström's method. So, our algorithm outperforms this algorithm for all values of σ^2.

In the second simulation, we compare the RMSE versus M/N of our proposed algorithm with the other methods. We varied M from 10 to 90 with a step of 5. We assume that $\sigma^2 = 0.2[m^2]$. The results are shown in Fig. 3. As can be seen, our algorithm outperforms the other MDS-like methods for all values of M/N.

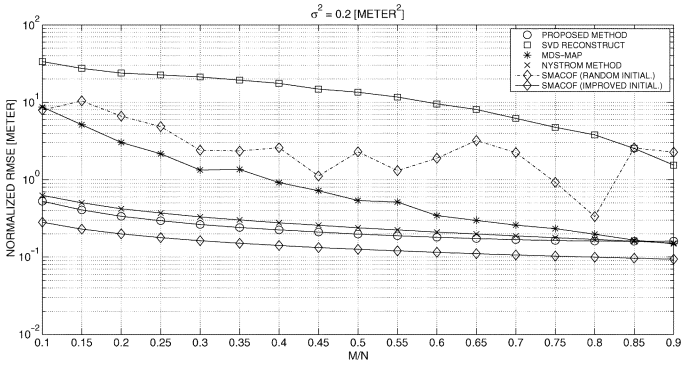

Fig. 3. RMSEs versus the ratio M/N for $\sigma^2 = 0.2$.

VI. Conclusions

We consider the problem of reconstructing the configuration of a sensor network (up to rotation, reflection, and translation) from pairwise distance measurements assuming the network is composed of two groups: one group contains nodes that communicate with each other, and the second group contains nodes that do not communicate with each other, and only communicate with each of the nodes in the first group. The classical MDS algorithm cannot be applied in this case. Our approach is based on performing two projections on the available set of distance measurements. Simulations show that the proposed algorithm outperforms other methods based on matrix completion techniques. Future work will focus on: 1) examining other projections and their effect on the node positions; 2) analyzing the performance of the first step; 3) evaluating the complexity of the proposed approach.

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